A unified formula for Steenrod operations in flag manifolds

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Abstract

The classical Schubert cells on a flag manifold G/H give a cell decomposition for G/H whose Kronecker duals (known as Schubert classes) form an additive base for the integral cohomology $H^*(G/H)$.

We present a formula that expresses Steenrod mod–p operations on Schubert classes in G/H in terms of Cartan numbers of G.

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1. Introduction

Let $p \geq 2$ be a fixed prime and let $\mathcal{A}_p$ be the mod–p Steenrod algebra. Denote by $\mathcal{P}^k \in \mathcal{A}_p$, $k \geq 0$, the Steenrod mod–p reduced powers on the $\mathbb{Z}_p$–cohomology of topological spaces [SE]. When $p = 2$, it is more customary to write $Sq^{2k} \in \mathcal{A}_2$ instead of $\mathcal{P}^k$.

In general, an entire description of the $\mathcal{A}_p$–action on the $\mathbb{Z}_p$–cohomology of a topological space X leads to two enquiries.

Problem A. *Specify an additive basis $S = \{\omega_1, \cdots, \omega_m\}$ for the graded $\mathbb{Z}_p$–vector space $H^*(X; \mathbb{Z}_p)$ that encodes the geometric formation of X (e.g. a cell decomposition of X).*

Problem B. *Determine all the coefficients $c_{k,i}^j \in Z_p$ in the expression*

$$P^k(\omega_i) = \sum_{1 \leq j \leq m} c_{k,i}^j \omega_j,\ k \geq 0,\ 1 \leq i \leq m.$$

The study of the internal structure of the algebra $\mathcal{A}_p$ has continued for almost fifty years, see Wood [W] for a thorough historical account, further problems and relevant references. On the other hand, we note that even partial solutions to Problem B can have significant consequences in manifold geometry. To mention only a few examples: the classical Wu–formula [Wu] can be interpreted as the expansion of the Sq^k–action on the special Schubert classes in the real Grassmannians; the calculation by Steenrod and Whitehead [SW] in the truncated real projective spaces led to an enormous step in understanding the classical problem of how many linearly independent vector fields can be found on the n-sphere S^n; by deriving partial knowledge of the $\mathcal{P}^k$–action on the special Schubert classes in the complex Grassmannian, Borel and Serre ([BSe$_1$,BSe$_2$]) demonstrated that the $2n$–dimensional spheres S^{2n} does not admit any almost complex structure unless $n = 1, 2, 3$. Needless to say many profound applications and deep implication of Steenrod operations in topology [D,St,Le], effective computation of these operations in given manifolds deserves also high priority.

Let G be a compact connected Lie group and let H be the centralizer of a one-parameter subgroup in G. The space $G/H = \{gH \mid g \in G\}$ of left cosets of H in G is known as a *flag manifold*. In this paper we study Problem A and B for all G/H .

Firstly, if X is a flag manifold G/H, a uniform solution to Problem A is already known as *Basis Theorem* from *the Schubert's enumerative calculus* (i.e. a branch of algebraic geometry [K, So, BGG]). It was originated by Ehresmann [E] for the Grassmannians $G_{n,k}$ of k-dimensional subspaces in $\mathbb{C}^n$ in 1934, extended to the case where G is a matrix group by Bruhat in 1954, and completed for all compact connected Lie groups by Chevalley [Ch] in 1958. We briefly recall the result.

Let W and W' be the Weyl groups of G and H respectively. The set W/W' of left cosets of W' in W can be identified with the subset of W:

$$\overline{W} = \{w \in W \mid l(w_1) \geq l(w) \text{ for all } w_1 \in wW'\},$$

where $l : W \to \mathbb{Z}$ is the length function relative to a fixed maximal torus T in G [BGG, 5.1. Proposition]. The key fact is that the space G/H admits a canonical decomposition into cells indexed by elements of $\overline{W}$

$$(1.1) \qquad G/H = \bigcup_{w\in\overline{W}} X_w, \quad \dim X_w = 2l(w),$$

with each cell X_w the closure of an algebraic affine space, known as a *Schubert variety* in G/H [Ch, BGG]. Since only even dimensional cells are involved in the decomposition (1.1), the set of fundamental classes $[X_w] \in H_{2l(w)}(G/H)$, $w \in \overline{W}$, forms an additive basis of the homology $H_*(G/H)$. The cocycle class $P_w(H) \in H^{2l(w)}(G/H)$ defined by the Kronecker pairing as

$$\langle P_w(H), [X_u]\rangle = \delta_{w,u},\ w, u \in \overline{W},$$

is called the *Schubert class corresponding to* w. Combining (1.1) with the Poincaré duality yields the following solution to Problem A.

Lemma 1 (Basis Theorem). *The set of Schubert classes* $\{P_w(H) \mid w \in \overline{W}\}$ *constitutes an additive basis for the ring* $H^*(G/H)$.

It follows from the Basis Theorem that, for a $u \in \overline{W}$ and $k \geq 0$, one has the expression

$$(1.2) \qquad \mathcal{P}^k(P_u(H)) \equiv \sum a^k_{w,u} P_w(H) \mod p,\ a^k_{w,u} \in \mathbb{Z}_p,$$

where the sum ranges over all $w \in \overline{W}$ with $l(w) = l(u) + k(p-1)$ for dimension reason. Thus, in the case of $X = G/H$, Problem B admits a concrete form.

> **Problem C.** *Determine the numbers* $a^k_{w,u} \in \mathbb{Z}_p$ *for* $k \geq 0$, $w, u \in \overline{W}$ *with* $l(w) = l(u) + k(p-1)$.

If G is the unitary group $U(n)$ of order n and $H = U(k) \times U(n-k)$, the flag manifold G/H is the complex Grassmannian $G_{n,k}$ of k-planes through the origin in $\mathbb{C}^n$. The i^{th} Chern classes $c_i \in H^{2i}(G_{n,k})$, $1 \leq i \leq k$, of the canonical complex k-bundle over $G_{n,k}$ are precisely *the special Schubert classes* on $G_{n,k}$ [So, GH]. In order to generalize the classical Wu–formula [Wu], many works were devoted to find an expression of $\mathcal{P}^k(c_i)$ in terms of the c_i (cf. [BSe$_1$, BSe$_2$, D$_4$, La, Le, P, S, Su]). This seems to be the only special case for which Problem C has been studied in some details.

It is well known that the knowledge of the $\mathcal{A}_p$–action on the $\mathbb{Z}_p$–cohomology of a space X can provide deeper information on the topology of X than just the cohomology ring structure. In this regard the present work is as a sequel of [D$_3$,DZ], where the multiplicative rule of Schubert classes (in the integral cohomology $H^*(G/H)$) was determined. In particular, in view of the geometric decomposition (1.1) of G/H offered by the classical Schubert cells, the numbers $a^k_{w,u}$ are immediately applicable to investigate the attaching maps of these cells (e.g. Compare the tables in Section 5 with the figure in [Le, Section 6]). We quote from Lenart [Le] for the case of $G_{n,k}$: *Apart from projective spaces, very little is known about the attaching maps of their cells.*

This paper is arranged as follows. Section 2 contains a brief introduction to the Weyl group associated with a Lie group. Then the solution to Problem C (i.e. the Theorem) is presented. After geometric preliminaries in Section 3 the Theorem is established in Section 4. In order to illustrate the effective computability of our method, computational results for some cases of G/H are explained and tabulated in the final section.

2. The result

To investigate a flag manifold G/H one may assume that the Lie group G under consideration is 1–connected and semi-simple ([BH]). Since all 1–connected semi–simple Lie groups are classified by their Cartan matrices [Hu, p.55], any numerical topological invariant of G/H may be reduced to the Cartan numbers (entries in the Cartan matrix of G). We present both a formula and an algorithm, that evaluate the numbers $a^k_{w,u} \in \mathbb{Z}_p$ in terms of Cartan numbers of G.

Fix a maximal torus T of G and set $n = \dim T$. Equip the Lie algebra $L(G)$ of G with an inner product $(,)$ so that the adjoint representation acts as isometries of $L(G)$. The *Cartan subalgebra* of G is the Euclidean subspace $L(T)$ of $L(G)$.

The restriction of the exponential map $\exp : L(G) \to G$ to $L(T)$ defines a set $D(G)$ of $\frac{1}{2}(\dim G - n)$ hyperplanes in $L(T)$, i.e. the set of *singular hyperplanes* through the origin in $L(T)$. The reflections σ of $L(T)$ in these planes generate the Weyl group W of G ([Hu, p.49]).

Fix a regular point $\alpha \in L(T) \setminus \cup_{L\in D(G)} L$ and let Δ be the set of simple roots relative to α [Hu, p.47]. If $\beta \in \Delta$ the reflection σ_β in the hyperplane $L_\beta \in D(G)$ relative to β is called a *simple reflection*. If $\beta, \beta' \in \Delta$, *the Cartan number*

$$\beta \circ \beta' = 2(\beta, \beta')/(\beta', \beta')$$

is always an integer (only $0, \pm 1, \pm 2, \pm 3$ can occur [Hu, p.55]).

It is known that the set of simple reflections $\{\sigma_\beta \mid \beta \in \Delta\}$ generates W. That is, any $w \in W$ admits a factorization of the form

$$w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m},\ \beta_i \in \Delta. \tag{2.1}$$

Definition 1. The *length* $l(w)$ of a $w \in W$ is the least number of factors in all decompositions of w in the form (2.1). The decomposition (2.1) is said *reduced* if $m = l(w)$.

If (2.1) is a reduced decomposition, the $m \times m$ (strictly upper triangular) matrix $A_w = (a_{i,j})$ with

$$a_{i,j} = \left\{ \begin{array}{c} 0 \text{ if } i \geq j; \\ -\beta_i \circ \beta_j \text{ if } i < j \end{array} \right.$$

will be called *the Cartan matrix of w* associated to the decomposition (2.1).

Let $\mathbb{Z}[x_1, \cdots, x_m] = \oplus_{n\geq 0} \mathbb{Z}[x_1, \cdots, x_m]^{(n)}$ be the ring of integral polynomials in $x_1, \cdots, x_m$, graded by $\mid x_i \mid = 1$.

Definition 2. For a subset $[i_1, \cdots, i_r] \subseteq [1, \cdots, m]$ and $1 \leq k \leq r$, denote by $m_{k,p}(x_{i_1}, \cdots, x_{i_r})$ the polynomial

$$\sum_{(\alpha_1,\cdots,\alpha_r)} x_{i_1}^{\alpha_1}\cdots x_{i_r}^{\alpha_r} \in \mathbb{Z}[x_1,\cdots,x_m]^{(r+k(p-1))},$$

where the sum is over all distinct permutations $(\alpha_1,\cdots,\alpha_r)$ of the partition $(p^k,1^{r-k})$ ([M, p.1]).

Remark 1. In the theory of symmetric functions, the $m_{k,p}(x_{i_1},\cdots,x_{i_r})$ is known as *the monomial symmetric function* in $x_{i_1},\cdots,x_{i_r}$ associated to the partition $(p^k,1^{r-k})$ ([M, p.19]). As examples, if $[i_1,\cdots,i_r]=[1,2,3]$ one has

$$\begin{aligned}
m_{1,p}(x_{i_1},\cdots,x_{i_r}) &= x_1^p x_2 x_3 + x_1 x_2^p x_3 + x_1 x_2 x_3^p;\\
m_{2,p}(x_{i_1},\cdots,x_{i_r}) &= x_1^p x_2^p x_3 + x_1 x_2^p x_3^p + x_1^p x_2 x_3^p;\\
m_{3,p}(x_{i_1},\cdots,x_{i_r}) &= x_1^p x_2^p x_3^p.
\end{aligned}$$

Definition 3. Given a $m\times m$ strictly upper triangular integer matrix $A=(a_{i,j})$ define a homomorphism $T_A:\mathbb{Z}[x_1,\cdots,x_m]^{(m)}\to\mathbb{Z}$ recursively as follows:

1) for $h\in\mathbb{Z}[x_1,\cdots,x_{m-1}]^{(m)}$, $T_A(h)=0$;

2) if $m=1$ (consequently $A=(0)$), then $T_A(x_1)=1$;

3) for $h\in\mathbb{Z}[x_1,\cdots,x_{m-1}]^{(m-r)}$ with $r\geq 1$,

$$T_A(hx_m^r)=T_{A'}(h(a_{1,m}x_1+\cdots+a_{m-1,m}x_{m-1})^{r-1}),$$

where A' is the $((m-1)\times(m-1)$ strictly upper triangular) matrix obtained from A by deleting the m^{th} column and the m^{th} row.

By additivity, T_A is defined for every $h\in\mathbb{Z}[x_1,\cdots,x_m]^{(m)}$ using the unique expansion $h=\sum_{0\leq r\leq m} h_r x_m^r$ with $h_r\in\mathbb{Z}[x_1,\cdots,x_{m-1}]^{(m-r)}$.

Remark 2. Definition 3 implies an effective algorithm to evaluate T_A.
For $k=2$ and $A_1=\begin{pmatrix}0 & a\\ 0 & 0\end{pmatrix}$, then $T_{A_1}:\mathbb{Z}[x_1,x_2]^{(2)}\to\mathbb{Z}$ is given by

$$\begin{aligned}
&T_{A_1}(x_1^2)=0,\\
&T_{A_1}(x_1x_2)=T_{A_1'}(x_1)=1 \text{ and}\\
&T_{A_1}(x_2^2)=T_{A_1'}(ax_1)=a.
\end{aligned}$$

For $k=3$ and $A_2=\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix}$, then $A_2'=A_1$ and $T_{A_2}:\mathbb{Z}[x_1,x_2,x_3]^{(3)}\to\mathbb{Z}$ is given by

$$T_{A_2}(x_1^{r_1}x_2^{r_2}x_3^{r_3}) = \{ \begin{array}{l} 0, \text{ if } r_3 = 0 \text{ and} \\ T_{A_1}(x_1^{r_1}x_2^{r_2}(bx_1+cx_2)^{r_3-1}), \text{ if } r_3 \geq 1, \end{array}$$

where $r_1 + r_2 + r_3 = 3$, and where T_{A_1} is calculated in the above.

Assume that $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m}$, $\beta_i \in \Delta$, is a reduced decomposition of $w \in \overline{W}$ and let $A_w = (a_{i,j})_{m\times m}$ be the associated Cartan matrix of w. For a subset $J = [i_1, \cdots, i_r] \subseteq [1, \cdots, m]$ we set

$$\sigma_J = \sigma_{\beta_{i_1}} \circ \cdots \circ \sigma_{\beta_{i_r}}.$$

Our solution to Problem C is

Theorem. *For a $u \in \overline{W}$, $k > 0$ with $l(u) = r$ and $m = r + k(p-1)$, we have (in (1.2)) that*

$$a_{w,u}^k \equiv T_{A_w}[\sum_{\substack{J=[i_1,\cdots,i_r]\subseteq[1,\cdots,m] \\ \sigma_J = u}} m_{k,p}(x_{i_1}, \cdots, x_{i_r})] \;\; mod\ p.$$

This result will be shown in Section 4.

Indeed, the Theorem indicates an effective algorithm to evaluate $a_{w,u}^k$ as the following recipe shows.

(1) Starting from the Cartan matrix of G, a program to enumerate all elements in $\overline{W}$ by their *minimal reduced decompositions* is available in [DZ], [DZZ];

(2) For a $w \in \overline{W}$ with a reduced decomposition, the corresponding Cartan matrix A_w can be read directly from Cartan matrix of G (Compare Definition 1 with [Hu, p.59]);

(3) For a $w \in \overline{W}$ with a reduced decomposition $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m}$ and a $u \in \overline{W}$ with $l(u) = r < m$, the solutions in the subsequence $J = [i_1, \cdots, i_r] \subseteq [1, \cdots, m]$ to the equation $\sigma_J = u$ in W agree with the solutions to the equation $\sigma_J(\alpha) = u(\alpha)$ in the vector space $L(T)$, where $\alpha \in L(T)$ is a fixed regular point;

(4) The evaluation the operator T_{A_w} on a polynomial can be easily programmed (cf. Definition 3 or [DZ, Section 5]).

Based on the algorithm explained above, a program to produce the numbers $a_{w,u}^k$ has been compiled. As examples some computational results from the program are explained and tabulated in Section 5.

3. Preliminaries in the K–cycles of Bott–Samelson

In this section all homologies (resp. cohomologies) will have integer coefficients unless otherwise specified. If $f : X \to Y$ is a continuous map between two topological spaces, f_* (resp. f^*) is the homology (resp. cohomology) map induced by f. If M is an oriented closed manifold (resp. a connected projective variety) $[M] \in H_{\dim M}(M)$ stands for the orientation class. The Kronecker pairing, between cohomology and homology of a space X, will be denoted by $<,>: H^*(X) \times H_*(X) \to \mathbb{Z}$.

The proof of our Theorem will make use of the celebrated K-cycles (i.e. *Bott-Samelson resolutions of Schubert varieties*) on the flag manifold G/T constructed by Bott and Samelson early in 1955 [BS_1]. In this Section we recall the construction of these cycles, as well as their basic properties (from Lemma 2 to Lemma 4) developed in [BS_1,BS_2,D_1,D_3]. The main technique result in this section is Lemma 5, which allows us to transform the proof of the Theorem for G/H to calculation in the K–cycles of Bott–Samelson.

As in Section 2, we fix a regular point $\alpha \in L(T)$ and let Δ be the set of simple roots relative to α. For a $\beta \in \Delta$, the singular plane in $L(T)$ relative to β will be denoted by L_β [Hu,p.47]. Write by K_β the centralizer of $\exp(L_\beta)$ in G, where exp is the restriction of the exponential map $L(G) \to G$ to $L(T)$. We note that $T \subset K_\beta$ and that the quotient manifold K_β/T is diffeomorphic to 2-sphere [BS_2, p.996].

The 2-sphere K_β/T carries a natural orientation $\omega_\beta \in H_2(K_\beta/T; \mathbb{Z})$ that may be specified as follows. The Cartan decomposition of the Lie algebra $L(K_\beta)$ relative to the maximal torus $T \subset K_\beta$ has the form $L(K_\beta) = L(T) \oplus \vartheta_\beta$, where $\vartheta_\beta \subset L(G)$ is a 2-plane, the *root space* belonging to the root β [Hu, p.35]. Let $[,]$ be the Lie bracket on $L(G)$. Take a non-zero vector $v \in \vartheta_\beta$ and let $v' \in \vartheta_\beta$ be such that $[v, v'] = \beta$. The ordered base $\{v, v\prime\}$ gives an orientation on ϑ_β which does not depend on the initial choice of v.

The tangential of the quotient map $\pi_\beta : K_\beta \to K_\beta/T$ at the group unit $e \in K_\beta$ maps ϑ_β isomorphically onto the tangent space to K_β/T at $\pi_\beta(e)$. In this manner the orientation $\{v, v\prime\}$ on ϑ_β furnishes K_β/T with the induced orientation $\omega_\beta = \{\pi_\beta(v), \pi_\beta(v\prime)\}$.

For a sequence $\beta_1, \cdots, \beta_m \in \Delta$ of simple roots (repetition like $\beta_i = \beta_j$ may occur) let $K(\beta_1, \cdots, \beta_m)$ be the product group $K_{\beta_1} \times \cdots \times K_{\beta_m}$. Since $T \subset K_{\beta_i}$ for each i the group $T \times \cdots \times T$ (m-copies) acts on $K(\beta_1, \cdots, \beta_m)$ from the right by

$$(g_1, \cdots, g_m)(t_1, \cdots, t_m) = (g_1 t_1, t_1^{-1} g_2 t_2, \cdots, t_{m-1}^{-1} g_m t_m).$$

Let $\Gamma(\beta_1, \cdots, \beta_m)$ be the base manifold of this principle action, oriented by the ω_{β_i}, $1 \leq i \leq m$. The point in $\Gamma(\beta_1, \cdots, \beta_m)$ corresponding to a $(g_1, \cdots, g_m) \in K(\beta_1, \cdots, \beta_m)$ is denoted by $[g_1, \cdots, g_m]$.

The integral cohomology of $\Gamma(\beta_1, \cdots, \beta_m)$ has been determined in [BS$_1$, Proposition II]. Let $\varphi_i : K_{\beta_i}/T \to \Gamma(\beta_1, \cdots, \beta_m)$ be the embedding induced by the inclusion $K_{\beta_i} \to K(\beta_1, \cdots, \beta_m)$ onto the i^{th} factor group, and put

$$y_i = \varphi_{i*}(\omega_{\beta_i}) \in H_2(\Gamma(\beta_1, \cdots, \beta_m)),\ 1 \leq i \leq m.$$

Form the $m \times m$ strictly upper triangular matrix $A = (a_{i,j})_{m\times m}$ by letting

$$a_{i,j} = \{ \begin{array}{c} 0 \text{ if } i \geq j; \\ -\beta_i \circ \beta_j \text{ if } i < j. \end{array}$$

Lemma 2 ([BS$_2$]). *The set $\{y_1, \cdots, y_m\}$ forms a basis for $H_2(\Gamma(\beta_1, \cdots, \beta_m))$. Further, let $x_i \in H^2(\Gamma(\beta_1, \cdots, \beta_m))$, $1 \leq i \leq m$, be the classes Kronecker dual to $y_1, \cdots, y_m$ as $< x_i, y_j >= \delta_{i,j}$, $1 \leq i, j \leq m$,then*

$$H^*(\Gamma(\beta_1, \cdots, \beta_m)) = \mathbb{Z}[x_1, \cdots, x_m]/I,$$

where I is the idea generated by

$$x_j^2 - \sum_{i<j} a_{i,j} x_i x_j,\ 1 \leq j \leq m.$$

In view of Lemma 2 we introduce an additive map $\int_{\Gamma(\beta_1,\cdots,\beta_m)} : \mathbb{Z}[x_1, \cdots, x_m]^{(m)} \to \mathbb{Z}$ by

$$\int_{\Gamma(\beta_1,\cdots,\beta_m)} h =< p_{\Gamma(\beta_1,\cdots,\beta_m)}(h), [\Gamma(\beta_1, \cdots, \beta_m)] >,$$

where $[\Gamma(\beta_1, \cdots, \beta_m)] \in H_{2m}(\Gamma(\beta_1, \cdots, \beta_m)) = \mathbb{Z}$ is the orientation class and where

$$p_{\Gamma(\beta_1,\cdots,\beta_m)} : \mathbb{Z}[x_1, \cdots, x_m] \to H^*(\Gamma(\beta_1, \cdots, \beta_m))$$

is the obvious quotient homomorphism. The geometric implication of the operator T_A in Definition 3 (Section 2) is seen from the next result.

Lemma 3 ([D$_1$, Proposition 2]). *We have*

$$\int_{\Gamma(\beta_1,\cdots,\beta_m)} = T_A : \mathbb{Z}[x_1, \cdots, x_m]^{(m)} \to \mathbb{Z}.$$

In particular, $\int_{\Gamma(\beta_1,\cdots,\beta_m)} x_1 \cdots x_m = 1$.

It follows also from Lemma 2 that the ring $H^*(\Gamma(\beta_1, \cdots, \beta_m))$ has the additive basis $\{x_{i_1} \cdots x_{i_r} \mid [i_1, \cdots, i_r] \subseteq [1, \cdots, m]\}$. Since the dimension of every x_i is 2, the action of the $\mathcal{P}^k \in \mathcal{A}_p$ on these base elements is determined by the Cartan formula [SE]. Let $m_{k,p}(x_{i_1}, \cdots, x_{i_r})$ be the monomial symmetric function in $x_{i_1}, \cdots, x_{i_r}$ associated to the partition $(p^k, 1^{r-k})$ (cf. Definition 2).

Lemma 4. *$\mathcal{P}^k(x_{i_1}\cdots x_{i_r}) \equiv m_{k,p}(x_{i_1},\cdots,x_{i_r})$ mod p.*

Let H be the centralizer of a one-parameter subgroup in G and let G/H be the flag manifold of left cosets of H in G. Assume (without loss the generality) that with respect to the fixed maximal torus $T \subset G$, $T \subseteq H \subset G$.

Definition 4. The map

$$\varphi_{\beta_1,\cdots,\beta_m;H} : \Gamma(\beta_1,\cdots,\beta_m) \to G/H$$

by $[g_1,\cdots,g_m] \to g_1\cdots g_m H$ is clearly well defined and will be called the *K-cycle* of Bott-Samelson on G/H associated to the sequence $\beta_1,\cdots,\beta_m \in \Delta$ of simple roots (cf. [D$_3$, Subsection 7.1]).

It was first shown by Hansen [H] in 1972 that, when $H = T$, certain K-cycles of Bott–Samelson provide disingulations of Schubert varieties in G/T. The following more general result allows us to bring the calculation of $\mathcal{P}^k$–action on the ring $H^*(G/H)$ (i.e. Problem C) to the computation of the action on the truncated polynomial algebra $H^*(\Gamma(\beta_1,\cdots,\beta_m))$, while the latter is easily handled by Lemma 4.

Lemma 5. *Let $\{P_w(H) \in H^{2l(w)}(G/H) \mid w \in \overline{W}\}$ be the set of Schubert classes on G/H (cf. Lemma 1). The induced cohomology ring map $\varphi^*_{\beta_1,\cdots,\beta_m;H}$ is given by*

$$\varphi^*_{\beta_1,\cdots,\beta_m;H}(P_w(H)) = (-1)^r \sum_{\substack{J=[i_1,\cdots,i_r]\subseteq[1,\cdots,m] \\ \sigma_J = w}} x_{i_1}\cdots x_{i_r},$$

where $r = l(w)$.

Proof. Since $T \subseteq H \subset G$, the map $\varphi_{\beta_1,\cdots,\beta_m;H}$ factors through $\varphi_{\beta_1,\cdots,\beta_m;T}$ in the fashion

$$\begin{array}{ccc} \Gamma(\beta_1,\cdots,\beta_m) & \overset{\varphi_{\beta_1,\cdots,\beta_m;T}}{\rightarrow} & G/T \\ & \searrow & \downarrow \pi \\ \varphi_{\beta_1,\cdots,\beta_m;H} & & G/H \end{array},$$

where π is the standard fibration with fiber H/T. By [D$_3$,Lemma 5.1] we have

(a) *Lemma 5 holds for the case $H = T$.*

From [BGG, §5] we find that

(b) *the induced map $\pi^* : H^*(G/H) \to H^*(G/T)$ is given by*

$$\pi^*(P_w(H)) = P_w(T),\ w \in \overline{W} \subset W.$$

Combining (a) and (b) verifies Lemma 5.□

4. Proof of the Theorem.

For a $u \in \overline{W}$ with $l(u) = r$ and a $k \geq 1$, we assume as in (1.2) that

$$(3.1) \qquad \mathcal{P}^k(P_u(H)) \equiv \sum_{l(v)=r+p(k-1), v\in\overline{W}} a^k_{v,u} P_v(H),\ a^k_{v,u} \in \mathbb{Z}_p.$$

Let $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_m}$, $\beta_i \in \Delta$, be a reduced decomposition of a $w \in \overline{W}$ with $m = r + k(p-1)$, and let $A_w = (a_{i,j})_{m\times m}$ be the associated Cartan matrix.

Let $\varphi_{\beta_1,\cdots,\beta_m;H} : \Gamma(\beta_1, \cdots, \beta_m) \to G/H$ be the K-cycle associated to the ordered sequence $(\beta_1, \cdots, \beta_m)$ of simple roots. Applying the ring map $\varphi^*_{\beta_1,\cdots,\beta_m;H}$ to the equation (3.1) in $H^*(G/H; \mathbb{Z}_p)$ yields in $H^*(\Gamma(\beta_1, \cdots, \beta_m); \mathbb{Z}_p)$ that

$$\begin{aligned} \varphi^*_{\beta_1,\cdots,\beta_m;H}\mathcal{P}^k(P_u(H)) &\equiv \varphi^*_{\beta_1,\cdots,\beta_m;H}[\sum_{l(v)=r+p(k-1)} a^k_{v,u}P_v(H)] \\ &\equiv (-1)^m a^k_{w,u} x_1 \cdots x_m, \end{aligned}$$

where the second equality follows from

$$\varphi^*_{\beta_1,\cdots,\beta_m;H}[P_v(H)] = \{ \begin{matrix} (-1)^m x_1 \cdots x_m \text{ if } v = w; \\ 0 \text{ if } v \neq w \end{matrix} .$$

by Lemma 5. On the other hand

$$\begin{aligned} &\varphi^*_{\beta_1,\cdots,\beta_m;H}\mathcal{P}^k(P_u(H)) \equiv \mathcal{P}^k\{\varphi^*_{\beta_1,\cdots,\beta_m;H}(P_u(H))\} \\ &\equiv (-1)^r \mathcal{P}^k\{\sum_{\substack{J=[i_1,\cdots,i_r]\subseteq[1,\cdots,m] \\ \sigma_J=u}} x_{i_1}\cdots x_{i_r}\} \qquad \text{(by Lemma 5)} \\ &\equiv (-1)^r \sum_{\substack{J=[i_1,\cdots,i_r]\subseteq[1,\cdots,m] \\ \sigma_J=u}} m_{k,p}(x_{i_1}, \cdots, x_{i_r}) \qquad \text{(by Lemma 4)}, \end{aligned}$$

where the first equality comes from the naturality of $\mathcal{P}^k$ [SE]. Summarizing, we get in $H^{2m}(\Gamma(\beta_1, \cdots, \beta_m); \mathbb{Z}_p) = \mathbb{Z}_p$ that

$$\sum_{\substack{J=[i_1,\cdots,i_r]\subseteq[1,\cdots,m] \\ \sigma_J=u}} m_{k,p}(x_{i_1}, \cdots, x_{i_r}) \equiv (-1)^{k(p-1)} a^k_{w,u} x_1 \cdots x_m.$$

Evaluating both sides on the orientation class $[\Gamma(\beta_1, \cdots, \beta_m)]$ *mod* p and noting that p is a prime, we get from Lemma 3 that

$$\begin{aligned} a^k_{w,u} &\equiv < \sum_{\substack{J=[i_1,\cdots,i_r]\subseteq[1,\cdots,m] \\ \sigma_J=u}} m_{k,p}(x_{i_1}, \cdots, x_{i_r}), [\Gamma(\beta_1, \cdots, \beta_m)] > \\ &\equiv T_{A_w}(\sum_{\substack{J=[i_1,\cdots,i_r]\subseteq[1,\cdots,m] \\ \sigma_J=u}} m_{k,p}(x_{i_1}, \cdots, x_{i_r})). \end{aligned}$$

This completes the proof.

5. Applications

The Theorem handles the Problem C in its natural generality in the sense that it applies uniformly to

(1) every flag manifold G/H;

(2) each Schubert classe in a given G/H; and

(3) is valid for every $k \geq 1$ and a prime $p \geq 2$.

Because of these reasons a single program can be composed to perform computation in various G/H (cf. the discussion at the end of Section 2). We list computational results from the program for some cases of G/H.

For the Lie groups G concerned below, we let a set of simple roots $\Delta = \{\beta_1, \cdots, \beta_n\}$ of G be given and ordered as that in [Hu, p.64-75].

For a centralizer H of a one-parameter subgroup in G, write $\overline{W}^r$ for the subset of $\overline{W} = W/W'$ consisting of the elements with length r (cf. Definition 1), where W (resp. W') is the Weyl group of G (resp. H). The set $\{P_w(H) \mid w \in \overline{W}^r\}$ forms a basis for the $2r$-dimensional cohomology $H^{2r}(G/H)$ by Lemma 1.

Let $\Delta_H \subset \Delta$ be the subset consisting of simple roots of H. Starting from the order on Δ as well as the subset $\Delta_H \subset \Delta$, programs to decompose each $w \in \overline{W}^r$ uniquely into a reduced product, called *the minimal reduced decomposition of* w, have been composed (cf. [DZZ,DZ]). If two w, $w' \in \overline{W}^r$ are given by their minimal reduced decompositions

$$w = \sigma_{\beta_{i_1}} \circ \cdots \circ \sigma_{\beta_{i_r}},\ w' = \sigma_{\beta_{j_1}} \circ \cdots \circ \sigma_{\beta_{j_r}},$$

we say $w < w'$ if there exists a $1 \leq d < r$ such that $(i_1, \cdots, i_d) = (j_1, \cdots, j_d)$, but $i_{d+1} < j_{d+1}$. With respect to this order $\overline{W}^r$ becomes an ordered set, hence can be written as $\overline{W}^r = \{w_{r,i} \mid 1 \leq i \leq \#\overline{W}^r\}$.

We write σ_i instead of σ_{β_i}, $\beta_i \in \Delta$. We list in Table A all $w_{r,i} \in \overline{W}$ by their minimal reduced decompositions; followed by Table B that expresses all nontrivial $\mathcal{P}^k(s_{r,i})$, where the notion $s_{r,i}$ is used to simplify the Schubert class $P_{w_{r,i}}(H)$ *mod p*.

Example 1. $G = G_2$ (the exceptional group of rank 2) and $H = T$ (a maximal torus).

Table A. *Elements of $\overline{W}$ and their minimal reduced decompositions.*

$w_{r,i}$	decomposition	$w_{r,i}$	decomposition	$w_{r,i}$	decomposition
$w_{1,1}$	σ_1	$w_{3,1}$	$\sigma_1\sigma_2\sigma_1$	$w_{5,1}$	$\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1$
$w_{1,2}$	σ_2	$w_{3,2}$	$\sigma_2\sigma_1\sigma_2$	$w_{5,2}$	$\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2$
$w_{2,1}$	$\sigma_1\sigma_2$	$w_{4,1}$	$\sigma_1\sigma_2\sigma_1\sigma_2$	$w_{6,1}$	$\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2$
$w_{2,2}$	$\sigma_2\sigma_1$	$w_{4,2}$	$\sigma_2\sigma_1\sigma_2\sigma_1$		

Table B. *Nontrivial $P^k(s_{r,i})$*

$s_{r,i}$	$\mathcal{P}^1(s_{r,i})$ $(p=3)$	$\mathcal{P}^1(s_{r,i})$ $(p=5)$
$s_{1,1}$	0	$3\,s_{5,1}$
$s_{1,2}$	$2\,s_{3,2}$	$2\,s_{5,2}$
$s_{2,1}$	$s_{4,1}$	0
$s_{2,2}$	$2\,s_{4,2}$	0
$s_{3,1}$	$s_{5,1}$	0

Example 2. $G = F_4$ (the exceptional group of rank 4) and $H = Spin(7) \times S^1$.

Table A. *Elements of $\overline{W}$*
and their minimal reduced decompositions.

$w_{r,i}$	decomposition	$w_{r,i}$	decomposition
$w_{1,1}$	σ_1	$w_{8,2}$	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{2,1}$	$\sigma_2\sigma_1$	$w_{9,1}$	$\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{3,1}$	$\sigma_3\sigma_2\sigma_1$	$w_{9,2}$	$\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{4,1}$	$\sigma_2\sigma_3\sigma_2\sigma_1$	$w_{10,1}$	$\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{4,2}$	$\sigma_4\sigma_3\sigma_2\sigma_1$	$w_{10,2}$	$\sigma_4\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{5,1}$	$\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$	$w_{11,1}$	$\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{5,2}$	$\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$	$w_{11,2}$	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{6,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$	$w_{12,1}$	$\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{6,2}$	$\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$	$w_{13,1}$	$\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{7,1}$	$\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$	$w_{14,1}$	$\sigma_2\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{7,2}$	$\sigma_2\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$	$w_{15,1}$	$\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$
$w_{8,1}$	$\sigma_1\sigma_2\sigma_3\sigma_2\sigma_4\sigma_3\sigma_2\sigma_1$		

Table B. *Nontrivial* $P^k(s_{r,i})$

$s_{r,i}$	$p=3$		$p=5$			$p=7$	
	$\mathcal{P}^1(s_{r,i})$	$\mathcal{P}^2(s_{r,i})$	$\mathcal{P}^1(s_{r,i})$	$\mathcal{P}^2(s_{r,i})$	$\mathcal{P}^3(s_{r,i})$	$\mathcal{P}^1(s_{r,i})$	$\mathcal{P}^2(s_{r,i})$
$s_{1,1}$	$s_{3,1}$	0	$s_{5,1}+2\,s_{5,2}$	0	0	$5\,s_{7,1}+2\,s_{7,2}$	0
$s_{2,1}$	$2\,s_{4,1}+2\,s_{4,2}$	$2\,s_{6,2}$	$s_{6,1}+4\,s_{6,2}$	$3s_{10,1}$ $+2s_{10,2}$	0	$4\,s_{8,1}+3\,s_{8,2}$	$s_{14,1}$
$s_{3,1}$	0	0	$s_{7,2}$	$4\,s_{11,2}$	$3\,s_{15,1}$	$s_{9,2}$	$3\,s_{15,1}$
$s_{4,1}$	$s_{6,2}$	0	$2\,s_{8,1}+4\,s_{8,2}$	0		$3\,s_{10,1}+3\,s_{10,2}$	
$s_{4,2}$	$s_{6,2}$	0	$4\,s_{8,1}+4\,s_{8,2}$	0		$3\,s_{10,1}+3\,s_{10,2}$	
$s_{5,1}$	$s_{7,1}$	0	$s_{9,1}+2\,s_{9,2}$	0		$5\,s_{11,2}$	
$s_{5,2}$	$2\,s_{7,2}$	0	$2\,s_{9,1}+4\,s_{9,2}$	0		$4\,s_{11,1}+3\,s_{11,2}$	
$s_{6,1}$	$2\,s_{8,1}+s_{8,2}$	$2s_{10,1}$ $+2s_{10,2}$	$3s_{10,1}+4s_{10,2}$	0		$3\,s_{12,1}$	
$s_{6,2}$	0	0	$2\,s_{10,1}$	0		$2\,s_{12,1}$	
$s_{7,1}$	0	0	$4s_{11,1}+3s_{11,2}$	0		$s_{13,1}$	
$s_{7,2}$	0	0	$3\,s_{11,2}$	0		$s_{13,1}$	
$s_{8,2}$	$s_{10,1}+s_{10,2}$	0	$2\,s_{12,1}$			$3\,s_{14,1}$	
$s_{9,1}$	$s_{11,1}+s_{11,2}$	0	$2\,s_{13,1}$			$2\,s_{15,1}$	
$s_{9,2}$	$s_{11,1}+2\,s_{11,2}$	0	0			$6\,s_{15,1}$	
$s_{10,1}$	$s_{12,1}$	$2\,s_{14,1}$	$4\,s_{14,1}$				
$s_{10,2}$	$2\,s_{12,1}$	$s_{14,1}$	$4\,s_{14,1}$				
$s_{11,2}$	0	0	$s_{15,1}$				
$s_{12,1}$	$s_{14,1}$						
$s_{13,1}$	$2\,s_{15,1}$						

Example 3. $G = SO(12)$ (the special orthogonal group of order 12) and $H = U(6)$. The flag manifold G/H is the Grassmannian of complex structures on the 12-dimensional real Euclidean space $\mathbb{R}^{12}$ [D$_2$].

Table A. *Elements of* $\overline{W}$ and their minimal reduced decompositions.

$w_{r,i}$	decomposition	$w_{r,i}$	decomposition
$w_{1,1}$	σ_6	$w_{8,2}$	$\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{2,1}$	$\sigma_4\sigma_6$	$w_{8,3}$	$\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{3,1}$	$\sigma_3\sigma_4\sigma_6$	$w_{9,1}$	$\sigma_1\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{3,2}$	$\sigma_5\sigma_4\sigma_6$	$w_{9,2}$	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{4,1}$	$\sigma_2\sigma_3\sigma_4\sigma_6$	$w_{9,3}$	$\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{4,2}$	$\sigma_3\sigma_5\sigma_4\sigma_6$	$w_{10,1}$	$\sigma_1\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{5,1}$	$\sigma_1\sigma_2\sigma_3\sigma_4\sigma_6$	$w_{10,2}$	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{5,2}$	$\sigma_2\sigma_3\sigma_5\sigma_4\sigma_6$	$w_{10,3}$	$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{5,3}$	$\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$	$w_{11,1}$	$\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{6,1}$	$\sigma_1\sigma_2\sigma_3\sigma_5\sigma_4\sigma_6$	$w_{11,2}$	$\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{6,2}$	$\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$	$w_{12,1}$	$\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{6,3}$	$\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$	$w_{12,2}$	$\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{7,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$	$w_{13,1}$	$\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{7,2}$	$\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$	$w_{14,1}$	$\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{7,3}$	$\sigma_3\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$	$w_{15,1}$	$\sigma_6\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$
$w_{8,1}$	$\sigma_1\sigma_2\sigma_6\sigma_4\sigma_3\sigma_5\sigma_4\sigma_6$		

Table B$_1$. *Nontrivial* $P^k(s_{r,i})$ *for* $p=3$

$s_{r,i}$	$\mathcal{P}^1(s_{r,i})$	$\mathcal{P}^2(s_{r,i})$	$\mathcal{P}^3(s_{r,i})$	$\mathcal{P}^4(s_{r,i})$	$\mathcal{P}^5(s_{r,i})$
$s_{1,1}$	$s_{3,1}+s_{3,2}$	0	0	0	0
$s_{2,1}$	$2\,s_{4,1}+s_{4,2}$	$s_{6,1}+2\,s_{6,2}+2\,s_{6,3}$	0	0	0
$s_{3,1}$	$s_{5,2}$	$2\,s_{7,1}+2\,s_{7,2}+2\,s_{7,3}$	$s_{9,1}+s_{9,2}+2s_{9,3}$	0	0
$s_{3,2}$	$2\,s_{5,2}$	$s_{7,1}+s_{7,2}+s_{7,3}$	$2s_{9,1}+s_{9,2}+s_{9,3}$	0	0
$s_{4,1}$	$s_{6,1}$	$2\,s_{8,1}+2\,s_{8,2}$	$2\,s_{10,1}+s_{10,2}$	$2\,s_{12,2}$	0
$s_{4,2}$	$s_{6,2}+s_{6,3}$	$2\,s_{8,1}+2\,s_{8,2}$	$2\,s_{10,1}+2\,s_{10,2}$	$s_{12,1}$	0
$s_{5,2}$	$s_{7,1}+s_{7,2}+s_{7,3}$	0	$s_{11,2}$	$s_{13,1}$	$2\,s_{15,1}$
$s_{5,3}$	$s_{7,2}+2\,s_{7,3}$	$s_{9,2}$	$s_{11,2}$	$s_{13,1}$	$2\,s_{15,1}$
$s_{6,1}$	$s_{8,1}+s_{8,2}$	0	0	0	
$s_{6,2}$	$s_{8,1}+s_{8,2}+s_{8,3}$	$2s_{10,1}+2s_{10,2}+2s_{10,3}$	$s_{12,1}+s_{12,2}$	0	
$s_{6,3}$	$2\,s_{8,3}$	$s_{10,1}+s_{10,2}+s_{10,3}$	$s_{12,1}+s_{12,2}$	0	
$s_{7,1}$	$s_{9,1}+2\,s_{9,2}$	$2\,s_{11,1}+2\,s_{11,2}$	0	0	
$s_{7,2}$	$s_{9,1}$	$2\,s_{11,1}+2\,s_{11,2}$	$2\,s_{13,1}$	$2\,s_{15,1}$	
$s_{7,3}$	$s_{9,1}+s_{9,2}$	$2\,s_{11,1}+2\,s_{11,2}$	0		
$s_{8,1}$	$2\,s_{10,2}$	$s_{12,1}+s_{12,2}$	$s_{14,1}$		
$s_{8,2}$	$s_{10,2}$	$2\,s_{12,1}+2\,s_{12,2}$	$2\,s_{14,1}$		
$s_{8,3}$	$s_{10,1}+s_{10,2}+s_{10,3}$	0	$s_{14,1}$		
$s_{9,1}$	$s_{11,1}+s_{11,2}$	0	0		
$s_{9,3}$	$s_{11,1}+s_{11,2}$	0	$s_{15,1}$		
$s_{10,1}$	$s_{12,1}+2\,s_{12,2}$	0			
$s_{10,2}$	$s_{12,1}+s_{12,2}$	0			
$s_{10,3}$	$s_{12,1}$	0			
$s_{11,1}$	$2\,s_{13,1}$	$s_{15,1}$			
$s_{11,2}$	$s_{13,1}$	$2\,s_{15,1}$			
$s_{13,1}$	$s_{15,1}$				

Table B$_2$. *Nontrivial $P^k(s_{r,i})$ for $p=5$*

$s_{r,i}$	$\mathcal{P}^1(s_{r,i})$	$\mathcal{P}^2(s_{r,i})$	$\mathcal{P}^3(s_{r,i})$
$s_{1,1}$	$s_{5,1}+3\,s_{5,2}+2\,s_{5,3}$	0	0
$s_{2,1}$	$3\,s_{6,1}+4\,s_{6,3}$	0	0
$s_{3,1}$	$4\,s_{7,2}+3\,s_{7,3}$	$2\,s_{11,1}$	$4\,s_{15,1}$
$s_{3,2}$	$2\,s_{7,1}+2\,s_{7,2}+2\,s_{7,3}$	$s_{11,1}$	$2\,s_{15,1}$
$s_{4,1}$	$4\,s_{8,1}+3\,s_{8,2}$	$s_{12,2}$	
$s_{4,2}$	$4\,s_{8,2}+4\,s_{8,3}$	$3\,s_{12,1}+2\,s_{12,2}$	
$s_{5,2}$	$2\,s_{9,1}+2\,s_{9,2}+\ 2\,s_{9,3}$	$2\,s_{13,1}$	
$s_{5,3}$	$2\,s_{9,1}+2\,s_{9,2}+\ 2\,s_{9,3}$	$2\,s_{13,1}$	
$s_{6,1}$	$2\,s_{10,1}+3\,s_{10,2}$	0	
$s_{6,2}$	$2\,s_{10,1}+2\,s_{10,2}+\ 4\,s_{10,3}$	$4\,s_{14,1}$	
$s_{6,3}$	$3\,s_{10,1}+2\,s_{10,2}+\ s_{10,3}$	0	
$s_{7,1}$	$4\,s_{11,1}$	$2\,s_{15,1}$	
$s_{7,2}$	$3\,s_{11,1}$	$4\,s_{15,1}$	
$s_{7,3}$	$4\,s_{11,1}$	$2\,s_{15,1}$	
$s_{8,1}$	$2\,s_{12,1}+2\,s_{12,2}$		
$s_{8,2}$	$4\,s_{12,1}+3\,s_{12,2}$		
$s_{8,3}$	$3\,s_{12,2}$		
$s_{9,1}$	$4\,s_{13,1}$		
$s_{9,3}$	$3\,s_{13,1}$		
$s_{10,1}$	$2\,s_{14,1}$		
$s_{10,2}$	$2\,s_{14,1}$		
$s_{11,1}$	$s_{15,1}$		
$s_{11,2}$	$4\,s_{15,1}$		

Example 4. $G=U(7)$ (the unitary group of order 7) and $H=U(3)\times U(4)$. The flag manifold G/H is the Grassmannian of 3-planes through the origin in $\mathbb{C}^7$.

Table A. *Elements of $\overline{W}$ and their minimal reduced decompositions.*

$w_{r,i}$	decomposition	$w_{r,i}$	decomposition	$w_{r,i}$	decomposition
$w_{1,1}$	σ_3	$w_{5,3}$	$\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$	$w_{8,1}$	$\sigma_1\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$
$w_{2,1}$	$\sigma_2\sigma_3$	$w_{5,4}$	$\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3$	$w_{8,2}$	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$
$w_{2,2}$	$\sigma_4\sigma_3$	$w_{6,1}$	$\sigma_1\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$	$w_{8,3}$	$\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3$
$w_{3,1}$	$\sigma_1\sigma_2\sigma_3$	$w_{6,2}$	$\sigma_1\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3$	$w_{8,4}$	$\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$
$w_{3,2}$	$\sigma_2\sigma_4\sigma_3$	$w_{6,3}$	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3$	$w_{9,1}$	$\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$
$w_{3,3}$	$\sigma_5\sigma_4\sigma_3$	$w_{6,4}$	$\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$	$w_{9,2}$	$\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$
$w_{4,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3$	$w_{6,5}$	$\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3$	$w_{9,3}$	$\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3$
$w_{4,2}$	$\sigma_2\sigma_5\sigma_4\sigma_3$	$w_{7,1}$	$\sigma_1\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$	$w_{10,1}$	$\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$
$w_{4,3}$	$\sigma_3\sigma_2\sigma_4\sigma_3$	$w_{7,2}$	$\sigma_1\sigma_4\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3$	$w_{10,2}$	$\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$
$w_{4,4}$	$\sigma_6\sigma_5\sigma_4\sigma_3$	$w_{7,3}$	$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_5\sigma_4\sigma_3$	$w_{11,1}$	$\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$
$w_{5,1}$	$\sigma_1\sigma_2\sigma_5\sigma_4\sigma_3$	$w_{7,4}$	$\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$	$w_{12,1}$	$\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3$
$w_{5,2}$	$\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3$				

Table B$_1$. *Nontrivial $P^k(s_{r,i})$ for $p=3$*

$s_{r,i}$	$\mathcal{P}^1(s_{r,i})$	$\mathcal{P}^2(s_{r,i})$	$\mathcal{P}^3(s_{r,i})$	$\mathcal{P}^4(s_{r,i})$
$s_{1,1}$	$s_{3,1}+2\,s_{3,2}+s_{3,3}$	0	0	0
$s_{2,1}$	$2\,s_{4,1}+s_{4,2}+2\,s_{4,3}$	$2\,s_{6,2}+s_{6,3}+s_{6,5}$	0	0
$s_{2,2}$	$s_{4,1}+2s_{4,2}+2s_{4,3}+2s_{4,4}$	$s_{6,1}+2\,s_{6,2}+s_{6,3}+\ s_{6,5}$	0	0
$s_{3,1}$	$s_{5,1}+2\,s_{5,2}$	$s_{7,2}+2\,s_{7,3}$	$s_{9,3}$	0
$s_{3,2}$	$2s_{5,1}+2s_{5,2}+2s_{5,3}+2s_{5,4}$	$2\,s_{7,1}+2\,s_{7,4}$	$s_{9,1}+2\,s_{9,2}+2\,s_{9,3}$	0
$s_{3,3}$	$s_{5,1}+2\,s_{5,3}+2\,s_{5,4}$	$2s_{7,1}+2s_{7,2}+s_{7,3}+2s_{7,4}$	$s_{9,1}+2\,s_{9,2}+s_{9,3}$	0
$s_{4,1}$	$2\,s_{6,1}+2\,s_{6,2}$	$s_{8,1}+2\,s_{8,2}+2\,s_{8,3}$	$s_{10,1}+s_{10,2}$	$s_{12,1}$
$s_{4,2}$	$2\,s_{6,1}+2\,s_{6,2}+2\,s_{6,4}$	$s_{8,1}+2\,s_{8,3}+s_{8,4}$	$2\,s_{10,2}$	$2s_{12,1}$
$s_{4,3}$	$2\,s_{6,2}+s_{6,3}+2\,s_{6,4}\ +s_{6,5}$	$2\,s_{8,1}+s_{8,2}+s_{8,4}$	$2\,s_{10,1}+s_{10,2}$	$s_{12,1}$
$s_{4,4}$	$s_{6,1}+2\,s_{6,4}$	$2\,s_{8,1}+s_{8,2}+s_{8,4}$	$2\,s_{10,1}+s_{10,2}$	$s_{12,1}$
$s_{5,1}$	$2\,s_{7,1}$	$s_{9,1}+2\,s_{9,2}$	0	
$s_{5,2}$	$2\,s_{7,1}+s_{7,2}+2\,s_{7,3}$	$s_{9,1}+2\,s_{9,2}$	0	
$s_{5,3}$	$2\,s_{7,1}$	$s_{9,1}+2\,s_{9,2}$	0	
$s_{5,4}$	$2s_{7,1}+2s_{7,2}+s_{7,3}+2s_{7,4}$	0	0	
$s_{6,2}$	$2\,s_{8,1}+2\,s_{8,2}+2\,s_{8,3}$	0	$2\,s_{12,1}$	
$s_{6,3}$	$2\,s_{8,2}+s_{8,3}$	$s_{10,1}$	0	
$s_{6,4}$	$2\,s_{8,1}+s_{8,2}+s_{8,4}$	0	$2\,s_{12,1}$	
$s_{6,5}$	$2\,s_{8,1}+s_{8,3}$	$2\,s_{10,1}$	$2\,s_{12,1}$	
$s_{7,1}$	$s_{9,1}+2\,s_{9,2}$	0		
$s_{7,2}$	$2\,s_{9,2}+2\,s_{9,3}$	$2\,s_{11,1}$		
$s_{7,3}$	$2\,s_{9,2}+2\,s_{9,3}$	$2\,s_{11,1}$		
$s_{7,4}$	$2\,s_{9,1}+s_{9,2}$	0		
$s_{8,1}$	$2\,s_{10,1}+2\,s_{10,2}$	$s_{12,1}$		
$s_{8,2}$	$s_{10,1}+2\,s_{10,2}$	$s_{12,1}$		
$s_{8,3}$	$2\,s_{10,2}$	$s_{12,1}$		
$s_{8,4}$	$s_{10,1}$			
$s_{9,1}$	$2\,s_{11,1}$			
$s_{9,2}$	$2\,s_{11,1}$			
$s_{10,2}$	$s_{12,1}$			

Table B_2. *Nontrivial $P^k(s_{r,i})$ for $p = 5$*

$s_{r,i}$	$\mathcal{P}^1(s_{r,i})$	$\mathcal{P}^2(s_{r,i})$
$s_{1,1}$	$s_{5,1} + 4\,s_{5,3}$	0
$s_{2,1}$	$4\,s_{6,1} + s_{6,2} + 4\,s_{6,4}$	$s_{10,1}$
$s_{2,2}$	$s_{6,1} + s_{6,2} + 4\,s_{6,4}$	$s_{10,1}$
$s_{3,1}$	$4\,s_{7,1} + s_{7,3}$	$s_{11,1}$
$s_{3,2}$	$s_{7,2} + s_{7,3} + 4\,s_{7,4}$	$2\,s_{11,1}$
$s_{3,3}$	$s_{7,1} + s_{7,2} + 4\,s_{7,4}$	$s_{11,1}$
$s_{4,1}$	$4\,s_{8,1} + s_{8,2} + s_{8,3}$	$s_{12,1}$
$s_{4,2}$	$s_{8,2} + s_{8,3} + 4\,s_{8,4}$	$2\,s_{12,1}$
$s_{4,3}$	$s_{8,1} + 4\,s_{8,2} + s_{8,3}$	$s_{12,1}$
$s_{4,4}$	$s_{8,1} + 4\,s_{8,4}$	$s_{12,1}$
$s_{5,1}$	$4\,s_{9,1} + s_{9,2}$	
$s_{5,2}$	$s_{9,2} + s_{9,3}$	
$s_{5,3}$	$4\,s_{9,1} + s_{9,2}$	
$s_{5,4}$	$s_{9,1} + s_{9,3}$	
$s_{6,2}$	$s_{10,1} + s_{10,2}$	
$s_{6,3}$	$2\,s_{10,2}$	
$s_{6,4}$	$4\,s_{10,1} + s_{10,2}$	
$s_{6,5}$	$s_{10,1} + 4\,s_{10,2}$	
$s_{7,2}$	$s_{11,1}$	
$s_{7,3}$	$2\,s_{11,1}$	
$s_{7,4}$	$4\,s_{11,1}$	
$s_{8,3}$	$2\,s_{12,1}$	
$s_{8,4}$	$3\,s_{12,1}$	

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